ON BAZILEVIČ AND CONVEX FUNCTIONS

BY
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1. **Introduction.** Let

(1) $$f(z) = \left\{\frac{\beta}{1+\alpha^2}\int_0^z (h(\zeta)-\alpha i)\zeta^{[-\alpha\beta i/(1+\alpha^2)]-1}g(\zeta)^{\beta/(1+\alpha^2)}\,d\zeta\right\}^{(1+\alpha i)/\beta}$$

where $h(z)=1+\sum_{n=1}^{\infty} c_n z^n$ satisfies $\operatorname{Re} h(z)>0$ in $|z|<1$, $g(z)$ is starlike in $|z|<1$, α is any real number and $\beta>0$.

Bazilevič [1], [8] introduced the above class of functions and showed that each such function is univalent in $|z|<1$.

Let $\alpha=0$ in (1). On differentiating we get

(2) $$zf'(z) = f(z)^{1-\beta}g(z)^{\beta}h(z)$$

and

(3) $$\operatorname{Re} h(z) = \operatorname{Re}\,(zf'(z)/f(z)^{1-\beta}g(z)^{\beta}) > 0 \quad \text{in } |z| < 1.$$

Thomas [12] called a function satisfying the condition (3) a Bazilevič function of type β.

Let $C(r)$ denote the curve which is the image of the circle $|z|=r<1$ under the mapping $w=f(z)$, and let $L(r)$ denote the length of $C(r)$. Let $M(r)=\max_{|z|=r}|f(z)|$.

Then Thomas [12] showed that if $f(z)$ is a Bazilevič function of type β, $0<\beta\leqq 1$, then

$$L(r) \leqq K(\beta)M(r)\log(1/(1-r)),$$

and if $M(r)\leqq(1-r)^{-\alpha}$ then

$$L(r) \leqq K(\alpha,\beta)(1-r)^{-\alpha} \quad \text{for } 0 < \alpha \leqq 2$$

where $K(\beta)$ and $K(\alpha,\beta)$ are constants depending only upon α and β.

2. **On Bazilevič functions.**

LEMMA 1. *Let $f(z)$ be mean p-valent in $|z|<1$. Then*

$$\int_0^r\int_0^{2\pi}\frac{|f'(\rho e^{i\theta})|^2}{|f(\rho e^{i\theta})|}\,\rho\,d\theta\,d\rho \leqq 2p\pi M(r) \quad \textit{for } 0 < r < 1.$$

We owe Lemma 1 to Hayman [3, p. 45].

THEOREM 1. *Let $f(z)$ be a Bazilevič function of type β and* $\arg f(z)$ *be a function of bounded variation on $|z|=\rho<1$.*

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Then we have

$$L(r) \leqq K(\beta)M(r)\log(1/(1-r)),$$

and also if $M(r) \leqq (1-r)^{-\alpha}$ *then* $L(r) \leqq K(\alpha,\beta)(1-r)^{-\alpha}$ *for* $0<\alpha\leqq 2$ *where* $K(\beta)$ *and* $K(\alpha,\beta)$ *are constants depending only upon* α *and* β.

Proof. We have

$$\begin{aligned} L(r) &= \int_0^{2\pi} |zf'(z)|\,d\theta = \int_0^{2\pi} |f(z)^{1-\beta}g(z)^\beta h(z)|\,d\theta \\ &\leqq \int_0^r\int_0^{2\pi} |(1-\beta)f'(z)f(z)^{-\beta}g(z)^\beta h(z)|\,d\theta\,d\rho \\ &\quad + \int_0^r\int_0^{2\pi} |f(z)^{1-\beta}\beta g'(z)g(z)^{\beta-1}h(z)|\,d\theta\,d\rho \\ &\quad + \int_0^r\int_0^{2\pi} |f(z)^{1-\beta}g(z)^\beta h'(z)|\,d\theta\,d\rho \\ &= J_1+J_2+J_3, \quad \text{say.} \end{aligned}$$

Applying (2) and Lemma 1 we obtain

$$J_1 = |1-\beta|\int_0^r\int_0^{2\pi}\left|\frac{zf'(z)^2}{f(z)}\right|d\theta\,d\rho \leqq 2\pi|1-\beta|M(r). \tag{4}$$

Since $g(z)$ is a starlike function we may write $zg'(z)=g(z)\phi(z)$, where $\phi(z) = 1+\sum_{n=1}^\infty d_nz^n$ and $\operatorname{Re}\phi(z)>0$ in $|z|<1$.

Then we have

$$\begin{aligned} J_2 &= |\beta|\int_0^r\int_0^{2\pi} |f'(z)\phi(z)|\,d\theta\,d\rho \\ &\leqq |\beta|\left\{\int_0^r\int_0^{2\pi}|f'(z)^2|\,d\theta\,d\rho\right\}^{1/2}\left\{\int_0^r\int_0^{2\pi}|\phi(z)^2|\,d\theta\,d\rho\right\}^{1/2}. \end{aligned}$$

It is well known [6], [10] that

$$\int_0^r\int_0^{2\pi}|f'(z)^2|\,d\theta\,d\rho \leqq \frac{2\pi M(r)^2}{r}$$

and

$$\int_0^r\int_0^{2\pi}|\phi(z)^2|\,d\theta\,d\rho \leqq 4\pi\log\frac{1+r}{1-r}.$$

Hence we have

$$J_2 \leqq 2|\beta|\sqrt{2\pi}\left(\frac{1}{r}\log\frac{1+r}{1-r}\right)^{1/2}M(r). \tag{5}$$

On the other hand it is well known [2] that

$$|h'(z)| \leqq 2\operatorname{Re}h(z)/(1-\rho^2) \quad \text{for } |z| \leqq \rho < 1.$$

Hence we have

$$\begin{aligned} J_3 &\leqq \int_0^r \int_0^{2\pi} |f(z)^{1-\beta}g(z)^\beta| \frac{2 \operatorname{Re} h(z)}{1-\rho^2}\, d\theta\, d\rho \\ &= 2 \operatorname{Re} \int_0^r \int_0^{2\pi} |f(z)^{1-\beta}g(z)^\beta| \frac{zf'(z)}{f(z)^{1-\beta}g(z)^\beta}\cdot\frac{1}{1-\rho^2}\, d\theta\, d\rho \\ &= 2 \operatorname{Re} \int_0^r \int_0^{2\pi} zf'(z) \exp\left(-i\{(1-\beta)\arg f(z)+\beta \arg g(z)\}\right) \frac{1}{1-\rho^2}\, d\theta\, d\rho. \end{aligned}$$

Now let us put $\Theta=(1-\beta)\arg f(z)+\beta \arg g(z)$. Then we have

$$\begin{aligned} \operatorname{Re} &\int_0^{2\pi} zf'(z) \exp\left(-i\{(1-\beta)\arg f(z)+\beta\arg g(z)\}\right) d\theta \\ &= \operatorname{Re} \int_0^{2\pi} \frac{1}{i}\cdot\frac{df(z)}{d\theta}\cdot e^{-i\Theta}\, d\theta = \operatorname{Re}\frac{1}{i}\int_{|z|=\rho} e^{-i\Theta}\frac{df(z)}{d\Theta}\, d\Theta \\ &= \operatorname{Re}\frac{1}{i}\left[e^{-i\Theta}f(z)\right]_{|z|=\rho} + \operatorname{Re}\int_{|z|=\rho} e^{-i\Theta} f(z)\, d\Theta \\ &= \operatorname{Re}\int_{|z|=\rho} e^{-i\Theta} f(z)\, d\Theta \\ &= \operatorname{Re}(1-\beta)\int_{|z|=\rho} e^{-i\Theta}f(z)\, d\arg f(z) + \operatorname{Re}\beta\int_{|z|=\rho} e^{-i\Theta} f(z)\, d\arg g(z) \\ &\leqq \operatorname{Re}(1-\beta)\int_{|z|=\rho} e^{-i\Theta} f(z)\, d\arg f(z) + 2\pi\beta M(r). \end{aligned}$$

Since $\arg f(z)$ is a function of bounded variation on $|z|=\rho<1$, there exists a bounded constant K and $\int_{|z|=\rho} |d\arg f(z)| \leqq 2\pi K$. Hence we have

(6)
$$\begin{aligned} J_3 &\leqq 2\pi\{|1-\beta|K+|\beta|\}M(r)\int_0^r \frac{1}{1-\rho^2}\, d\rho \\ &= \pi\{|1-\beta|K+|\beta|\}M(r)\log\frac{1+r}{1-r}. \end{aligned}$$

From (4), (5) and (6) we obtain

$$L(r) \leqq K(\beta)M(r)\log(1/(1-r)).$$

Applying the same method as the above we have also that if $M(r)\leqq(1-r)^{-\alpha}$, then $L(r)\leqq K(\alpha,\beta)(1-r)^{-\alpha}$ for $0<\alpha\leqq 2$. This completes our proof.

COROLLARY 1. *Let $f(z)=z+\sum_{n=2}^\infty a_nz^n$ be regular and starlike in $|z|<1$. Then we have $L(r)\leqq KM(r)\log(1/(1-r))$, and also if $M(r)\leqq(1-r)^{-\alpha}$ then $L(r)\leqq K(\alpha)(1-r)^{-\alpha}$ for $0<\alpha\leqq 2$, where K is a bounded constant and $K(\alpha)$ is a constant depending only upon α.*

These results are due to Keogh and Pommerenke [4], [5].

LEMMA 2. *Let* $f(z)=\sum_{n=1}^{\infty} a_n z^n$ *be regular and univalent in* $|z|<1$.
Then we have

$$M(r) \leqq 4(\pi^{-1}A(r)\log(3/(1-r)))^{1/2}$$

where $A(r)$ *is the area enclosed by the curve* $C(r)$.

We owe this lemma to Pommerenke [7].

Applying Lemma 2 and the same method as in the proof of Theorem 1 we easily obtain the following theorems.

THEOREM 2. *Let* $f(z)$ *be a Bazilevič function of type* β *and* $\arg f(z)$ *be a function of bounded variation on* $|z|=\rho<1$.
Then we have

$$L(r) \leqq K(\beta)(A(r))^{1/2}(\log(1/(1-r)))^{3/2}$$

where $K(\beta)$ *is a constant depending only upon* β.

THEOREM 3. *Let* $f(z)$ *be regular and close-to-convex* (*a Bazilevič function of type* 1) *in* $|z|<1$. *Then we have*

$$L(r) \leqq K(A(r))^{1/2}(\log(1/(1-r)))^{3/2}$$

where K is a bounded constant.

REMARK. Thomas [11] has obtained the following:
Let $f(z)$ be regular and starlike in $|z|<1$. Then

$$L(r) \leqq 2(\pi A(r))^{1/2}\left(1+\log\frac{1+r}{1-r}\right).$$

3. On convex functions.

LEMMA 3. *Let* $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ *be regular and starlike in* $|z|<1$.
Then we have for any real λ

$$\int_0^{2\pi} |f(re^{i\theta})|^{\lambda}\, d\theta \leqq \int_0^{2\pi} |1+re^{i\theta}|^{-2\lambda}\, d\theta.$$

We owe this lemma to Robertson [9].

THEOREM 4. *Let* $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ *be regular and convex in* $|z|<1$.
Then we have

$$I(r) = \int_0^{2\pi} |f(z)|\, d\theta \leqq K\log\frac{1}{1-r}$$

where K is a bounded positive constant.

Proof. Since $f(z)$ is convex it follows that $zf'(z)$ is starlike in $|z|<1$. Then we have the following inequalities for a fixed constant $0<r_1<r<1$,

$$I(r) \leqq \int_0^r \int_0^{2\pi} |f'(z)|\, d\theta\, d\rho = \int_0^{r_1} \int_0^{2\pi} |f'(z)|\, d\theta\, d\rho + \int_{r_1}^r \int_0^{2\pi} |f'(z)|\, d\theta\, d\rho$$
$$\leqq C + \frac{1}{r_1} \int_{r_1}^r \int_0^{2\pi} |zf'(z)|\, d\theta\, d\rho,$$

where C is a bounded constant.

Applying Lemma 3 to $zf'(z)$ we have

$$\int_{r_1}^r \int_0^{2\pi} |zf'(z)|\, d\theta\, d\rho \leqq \int_{r_1}^r \int_0^{2\pi} \frac{1}{|1-\rho e^{i\theta}|^2}\, d\theta\, d\rho$$
$$\leqq \int_0^r \frac{1}{1-\rho^2}\, d\rho = \frac{1}{2} \log \frac{1+r}{1-r}.$$

Hence we have

$$I(r) \leqq C + \frac{1}{2r_1} \log \frac{1+r}{1-r}.$$

Integrating the following convex function we have

$$\int_{|z|=r} \left| \frac{z}{1-z} \right| d\theta = \int_0^{2\pi} \frac{r}{(1-2r\cos\theta + r^2)^{1/2}}\, d\theta = O(\log(1/(1-r))) \tag{7}$$

where O in (7) cannot be replaced by o.

This shows that the function $f(z)=z/(1-z)$ is an extremal function in Theorem 4.

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